CP–75–8

ANALYSIS AND COMPUTATION OF EQUILIBRIA AND REGIONS OF STABILITY

With Applications in Chemistry, Climatology, Ecology, and Economics

RECORD OF A WORKSHOP

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Stability versus Resilience

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Abstract

The complexity of biological systems (their structural hierarchy) points out dynamically the time-scales hierarchy of different processes.

This leads to the hierarchy in the motion of stability.

Rapid processes do not have full y stable, but "metastable" states and evolve slowly into the exact equilibrium state.

The notion "lability" (well known in medicine) reflects the existence of the surface of the possible quasi-equilibrium states.

Resilience (in my opinion) is only a particular case (and not defined rigorously) of the notion "adaptivity" (also well known). But the definition of adaptation follows from the idea of stability, which is presented more precisely below.

Hierarchy of Time Scales

I. Rapid and Slow Variables

Correct models of biological (ecological, in particular) systems usually contain a small parameter ε

$$\frac{d\vec{x}}{dt} = \varepsilon\vec{a}(\vec{x},\vec{y})$$

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{x},\vec{y}) + \varepsilon\vec{b}(\vec{x},\vec{y}) \qquad (1)$$

where ε is the ratio of characteristic times

$$\varepsilon = \frac{\text{"y-time"}}{\text{"x-time"}} \quad . \qquad (2)$$

Rigorous study of such systems was begun in the well-known work of A.N. Tikhonov. The trajectories structure can be found supposing that $\varepsilon = 0$

$$\frac{d\vec{\mathbf{x}}}{dt} = 0$$

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{x},\vec{y}) \quad . \tag{3}$$

In the new system vector $\vec{x}$ is constant,

$$\vec{x} = \vec{\text{constant}} = \vec{\alpha} \tag{4}$$

and, consequently, performs the role of a parameter

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{\alpha},y) \quad . \tag{5}$$

Stationary state of this system are found from the equation

$$\vec{f}(\vec{\alpha},\vec{y}) = 0 \quad .$$

Let us consider the simplest possiblity when $\vec{y}$ and $\vec{\alpha}$ are scalars. Even in this case, the set of stationary states is not a discrete collection of isolated fixed-points but a continuous curve on the surface (α,y).

Just the rich structure of equilibrium (to be more exact, quasi-equilibrium) state sets determines the complexity of the stability area concept.

For instance, let the equation $f(\alpha,y) = 0$ have a number of solutions with different α,

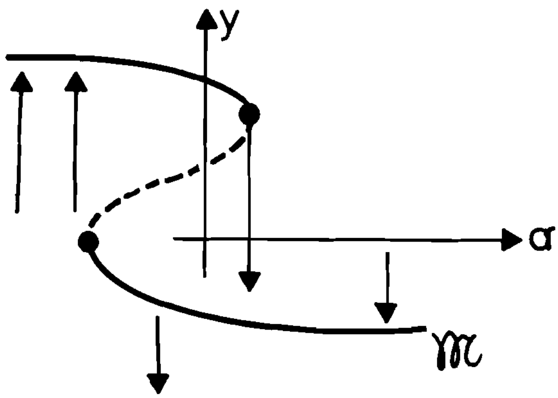


Figure 1.

Two semi-infinite branches of stable quasi-equilibria are connected by the arc of unstable ones. The upper branch corresponds to the "working-state" of the system, the lower branch describes the possibility of hysteresis.

II. The Evolution System

Now remember that the system (5) approximately describes a complete system. We shall bring back a slow evolutional motion. In our simplest case it is enough to solve a quasi-equilibrium equation

$$f(x,y) = 0 \quad , \tag{7}$$

$$y = \gamma(x)$$

and to introduce y into the first equation of the exact system

$$\frac{dx}{dt} = \varepsilon\ \alpha(x,\gamma(x)) \quad .$$

One should keep in mind that evolution takes place in a different way on each of the branches of the quasi-equilibrium curve

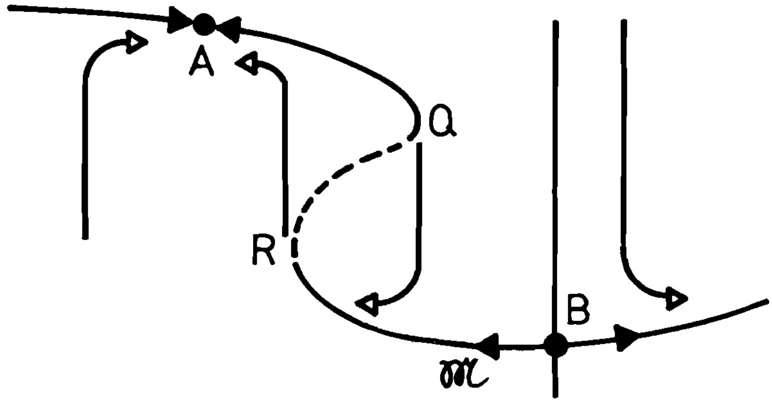


Figure 2.

with approximate equilibrium on the curve M of metastable states. On the upper brance the exact state (A) is stable, on the lower one it is unstable (B).

III. External Perturbations

Let us analyze the situation depicted on Figure 2 in a more detailed manner and show that knowing the curve M and the points A, B, Q on it fully determines the system behavior with regard to perturbations.

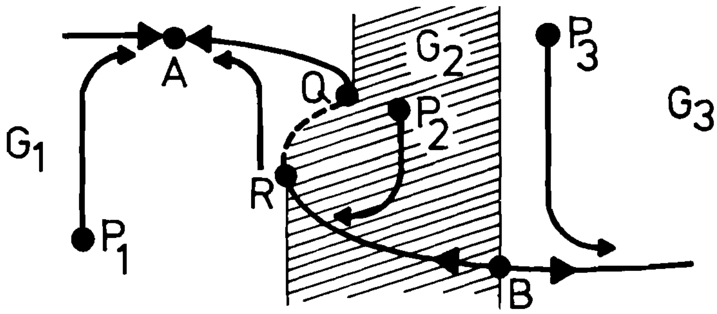


Figure 3. Splitting of the curve M into three branches: AQ_1 is a metastable brance, RB an adaptive brance and BB^1 an unstable branch.

If the perturbation moves the system to any point of G_1 area then its future will be the same as for the point P_1.

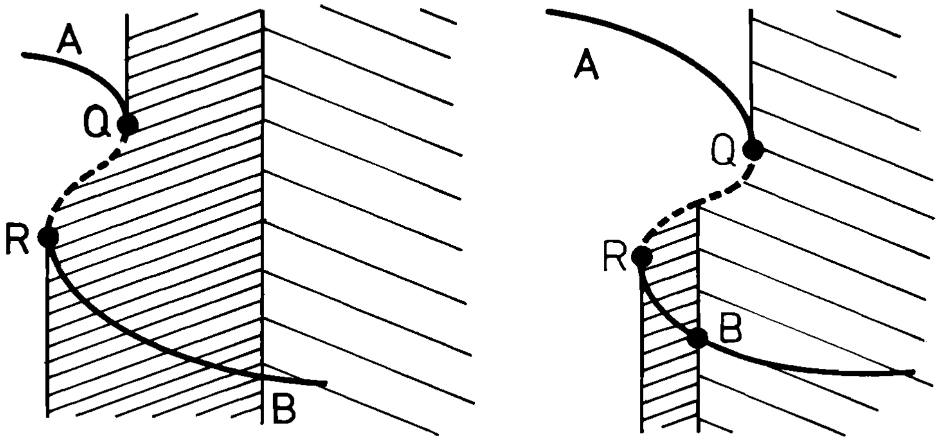
The system will rapidly move to the upper "work-branch" of the curve M and will slowly return to the exact state of equilibrium.

Point P_2 is a typical representative of the G_2 area. The events take a different turn. The system rapidly falls into the "shock-state", remains in it for a long time (while the evolution $R_2 \to R$ is taking place), but in the long run, "collects its strength" and returns to the "work branch" and only to the point S. Then slow restoration takes place, that is evolution $S \to A$.

Coming to G_3 area means death of the system if we mean by this the impossiblity to return to the work state that is on the upper branch.

IV. Stability and Adaptivity

Mutual disposition of the points A and B on the curve M can greatly change the character of the system reaction to perturbation.



weakly stable but greatly adaptive system	greatly stable, practically non-adaptive system

Comparing these two figures reveals an important difference between adaptive and stable figures.

The adaptive system "falls into a shock state" already in case of small perturbations but is able "to recover" even after strong shocks.

The stable system without adaptation on the contrary, preserves its "working ability" even in case of great perturbations but falling into a shock state almost means death for it.

V. Resumé

The concept of resilience is related to the concept of metastability of a rapid motion rather than to the traditional concept of stability. The contrasting of resilience and stability arises if slowly changing variables are treated as parameters but everything implicitly takes its own place if evolutionary system variations are taken into consideration.

A general stability concept may reasonably be specialized for systems with a time-scale hierarchy. The concept of metastability may be reserved for describing stability of cut down systems of rapid motions.

A stability of slow motions may be described by the well-known term "adaptivity."

The term "resilience" may be used as a synonym of adaptivity for the particular case of ecological systems.

Such an approach allows for a rigorous definition of the resilience concept which previously was introduced at an intuitive level.